Validation of the VirtuCath Stiffness Model

A wire-level finite-element model and a fast trained calculator for the bending, torsional and axial stiffness of reinforced shafts

VirtuCath LLC | Version 1.0 | October 2026

Summary

Catheter design needs the stiffness of each shaft section before anyone builds a sample. VirtuCath calculates it with a **trained calculator** that gives the bending, torsional and axial stiffness of a section in milliseconds. The calculator learned from 2,228 solutions of a **wire-level finite-element (FEM) model**, which places every wire on its real path through the polymer wall and uses datasheet materials. **No constant in either model was fitted to bench data.** These three stiffnesses are also the main inputs to VirtuCath's other models, such as steering, kink and pushability, so their accuracy carries into every result built on them.

Both models were checked following the ASME standards for computational models (V&V 10, 20 and 40). The main results are:

7 of 7	6.9 %	0	2,228
trusted bench comparisons inside the plan limits	median error of the calculator against the bench	bench-fitted constants	FEM solutions behind the calculator

- **Against measurements:** on seven trusted bench measurements of published and third-party shafts, the calculator's median error is 6.9 % (largest 17.7 %), inside the limits set in the validation plan before testing: 15 % (median) and 25 % (largest).
- **On new designs:** on a sealed test of designs that nobody looked at before the calculator was frozen, its median error is 7.9 % in bending, 8.3 % in torsion and 2.6 % in axial stiffness.
- **Where it is strongest:** single braids and steerable shafts. **Where to take more care:** the bending stiffness of a coil under a braid, where the median error is 18.4 %.
- **Comparing designs:** even when a design's absolute value is off, the difference between two similar designs is predicted much more closely: a median error of 2.4 % in bending, 1.6 % in torsion and 0.8 % in axial stiffness for the ratio between two designs, against 9.7 %, 7.7 % and 2.4 % for each design on its own.

A blind bench campaign, with shafts built and measured without access to the predictions, is planned as a future step (Section 6).

Figure 1. Three shaft construction examples, shown with the wire geometry of the FEM model. The outer polymer layers are cut back. (a) Braid of 16 ribbon wires over a liner. (b) Ribbon coil under a 24-wire braid. (c) Triaxial braid: a braid with straight axial wires. The colours of the polymer layers are for legibility only.

1 Why shaft stiffness is hard to predict

The steering, push and kink behaviour of a catheter depends on the stiffness of each shaft section: the bending stiffness *EI*, the torsional stiffness *GJ* and the axial stiffness *EA*. A shaft section is a polymer wall with metal reinforcement: braids, coils, and lumens for pull wires (Fig. 1).

Classical laminate theory (CLT) [7] treats the reinforcement as smeared plies of continuous fibre in a matrix of comparable stiffness. A catheter wall is different in two ways:

- the wires cross, press on each other, and can stick or slide where they cross;
- the polymer is 100 to 10,000 times softer than the wire, so the way the wires engage sets much of the bending and torsion response.

Laminate theory alone therefore cannot give these values. VirtuCath resolves the wire mechanics in a FEM model and carries that knowledge into a fast calculator, as the next section describes.

2 The approach

FEM model. The model places each braid and coil wire on its real interlaced path through the polymer wall (Fig. 2). Wires can stick, scissor or slide where they cross, with friction and stick-slip behaviour at the crossings. The model solves one repeating length of the section and returns *EI*, *GJ* and *EA*. The headline values are at the curvature and twist of a standard bench test. The model represents the shaft as designed and perfectly made: the polymer fills every gap. All material properties are datasheet values.

Trained calculator. The calculator keeps the familiar layer-by-layer structure of the base CLT model, but it learns its correction factors from FEM solutions (Table 1). It gives a result in milliseconds, so it runs inside VirtuCath.

Intended use. The models support early and mid design: screening and comparing constructions. VirtuCath does not replace design verification tests.

How much evidence is enough. The ASME V&V 40 standard [3, 4] sets the evidence a model needs from two questions: how much a decision rests on the model (*model influence*), and what a wrong decision costs (*decision consequence*). Here the model guides design choices next to testing (medium influence), and a wrong answer costs a build iteration, not patient harm (low to medium consequence). That calls for a *medium* credibility goal: the code is shown to solve its equations correctly, its numerical error is shown to be small, and it is compared with independent theory and with measurements. A model that replaced testing, or that a safety decision rested on, would need more, for example blind validation on many built shafts.

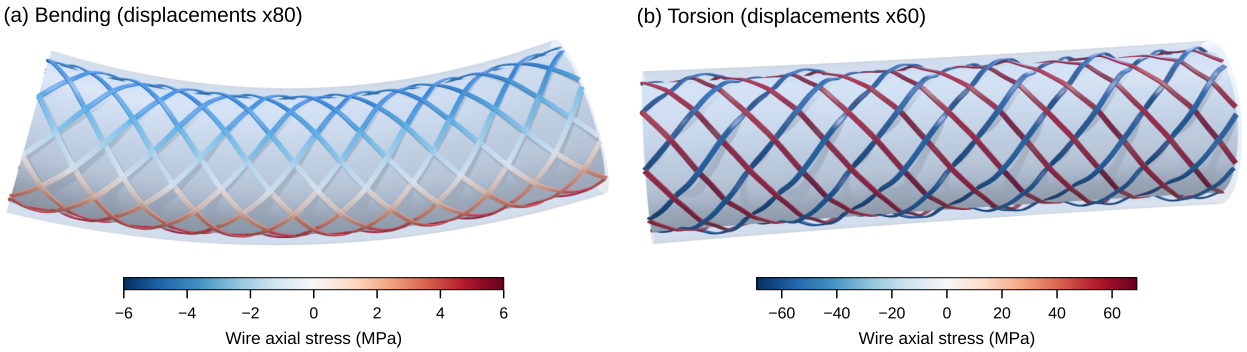

Figure 2. FEM solution for the braided shaft of Fig. 1a in bending (a) and in torsion (b). Wires are coloured by axial stress. In bending, the wires on the tension side stretch and those on the compression side shorten. In torsion, one helix family is in tension and the other in compression. Displacements are magnified; the outer polymer is hidden.

Table 1. FEM solutions used to train and test the calculator.

Construction family	Designs
Single braid	947
Single braid with pull-wire lumens (steerable)	486
Coil only	330
Coil under braid	357
Two overlapping braids	55
Triaxial braid	53
Total	**2,228**

3 How the models were checked

The checks follow the ASME standards for verification and validation of computational models [1, 2] and answer six questions, one after the other (Fig. 3). The acceptance criterion for each check was written before the check was run. Table 2 gives the answers.

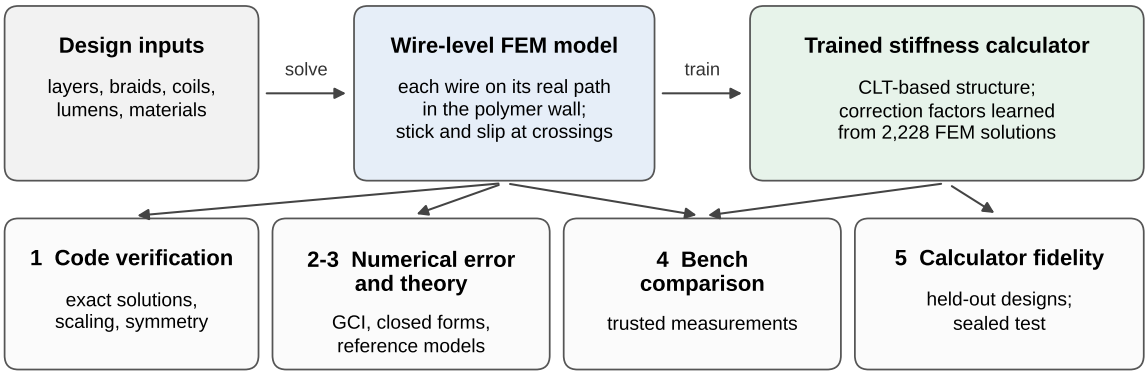

Figure 3. Structure of the evidence. The FEM model solves each design, and the trained calculator learns from the FEM solutions. Verification and benchmarks check the FEM model. The bench comparison checks both models against measurements. The fidelity tests check the calculator against the FEM model on designs that it did not learn from.

Table 2. The evidence at a glance.

Question	How it was checked	Result	Status
1. Does the code solve the equations correctly?	Exact solutions, scaling laws, symmetry and limit cases	All within tolerance	**Pass**
2. Is the numerical error small?	Grid convergence index on 36 designs, 483 runs	Median 0.1–2 %	**Pass**
3. Does it agree with independent theory?	Six benchmarks against theory and independent models	0–5 % (one bending-dominated case 10–15 %)	**Pass**
4. Does it agree with measurements?	Seven trusted bench comparisons	Median 8.8 % (FEM), 6.9 % (calculator)	**Pass**
5. Does the calculator reproduce the FEM model?	1,433 designs it did not learn from, four kinds of split	Median 8.7 / 6.6 / 2.1 % (*EI* / *GJ* / *EA*)	**Pass**
6. Does the frozen calculator hold up on a blind test?	Sealed test of 400 designs, scored once after the freeze	Median 7.9 / 8.3 / 2.6 %	**Done**
Blind bench check	Shafts built and measured without access to the predictions	–	Future work

4 Results

4.1 The code solves the equations correctly

The FEM model passed all exact checks within their tolerances (Table 3).

Table 3. Code verification against exact solutions.

Check	Result	Criterion
Multilayer tube, closed form	$< 10^{-6}$	10^{-6}
Thin tube against laminate theory	$< 0.5\,\%$	$0.5\,\%$
Length similarity ($\times 2$)	4×10^{-4}	$1\,\%$
Modulus scaling ($\times 3$)	8×10^{-7}	10^{-6}
Solved length doubled	1.3×10^{-5}	$0.5\,\%$
Order of the contact limits	holds	$0.1\,\%$
Left- and right-handed coil mirror	within $1\,\%$	$1\,\%$
Stiffness matrix symmetric and positive definite	yes	–

4.2 The numerical error is small

Each resolution setting of the model was refined in three levels on 36 designs (six families $\times$ six designs), and the grid convergence index (GCI) was calculated [5, 6]. The median GCI was 0.1–2 % for every setting and every stiffness (Fig. 4). Only one value touched the limit: the torsion GCI for the nodes per coil turn had a median of 2.01 %, 0.01 point above the 2 % criterion.

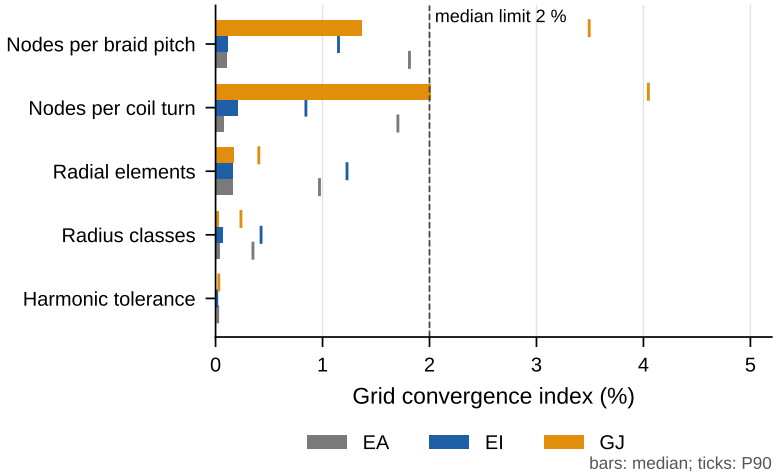

Figure 4. Grid convergence index for each resolution setting on 36 designs. Bars: median; ticks: P90 (the value that 90 % of designs stay below). Dashed line: the limit for the median.

> **In short.** Numerical error is not a significant part of the model's error. It is much smaller than the differences seen against the bench.

4.3 The model agrees with independent theory

Where theory or an independent reference model exists, the FEM model agreed within 0–5 % (Table 4). A welded wire net in torsion, a bending-dominated case, agreed with an independent 3-D frame model within 10–15 %.

Table 4. Benchmarks of the FEM model against independent theory or reference models.

Benchmark	Reference	Result	Verdict
Bare helical spring, 5–75°, round and ribbon wire	Kirchhoff rod theory [9, 10]	$< 10^{-6}$ converged; -0.7 to -1.1 % at production resolution	Pass
Wall with off-centre lumens, torsion	Exact Saint-Venant solution, two methods [11]	0.999–1.010	Pass
Thin composite plates	Laminate theory with an exact ply [7]	*EA*, *EI* within 0.4 %	Pass
Ply reference cell	Hashin–Rosen and Rayleigh closed forms [12, 13]	within 0.1 %	Pass
Wire-to-wall load transfer, four shafts	Independent finite-element model of each layer stack	0.95–1.01	Pass
Welded two-family wire net, torsion	Independent 3-D frame model	within 10–15 %	Bending-dominated

4.4 Both models agree with the bench

Bench data. Two kinds of bench data were used (Fig. 5):

- **Trusted data** set the plan limits: three published shafts (P1–P3), one 7 Fr design built in three braid patterns [8], with measured *EI* and *GJ*; and one third-party coil-under-braid shaft (T1) with measured *EA*.
- **In-house data** (H1–H5) come from a three-point-bend rig. They show direction only.

The limits were set in the validation plan before the final evaluation: a median error of 15 % or less, and no error above 25 %, over the seven trusted comparisons.

Result. All seven trusted comparisons were inside the plan limits for both models (Table 5). The FEM model had a median error of 8.8 % (largest 20.7 %). The trained calculator had a median of 6.9 % (largest 17.7 %). The calculator does not learn from the bench, so its small advantage on these shafts is partly chance.

The largest error is the torsion of the diamond-pattern shaft P3 (-20.7 % for the FEM model). This shaft is a sparse net of four wire pairs in each direction, and its torsion depends on details of the joints and the crimp. The published bench values are averages; their scatter is not reported, so this comparison does not include bench uncertainty.

In-house shafts. H1–H3 agree with the FEM model within -22 to $+1$ %. Two shafts (H4, H5) have a complex geometry that the models represent as an approximation, so only their ratio is compared; it agrees within 1 %. Two further in-house shafts gave inconclusive results and are being retested; they will be added in a later edition of this report.

> **In short.** Without any bench tuning, both models meet the plan limits on every trusted measurement.

Table 5. Trusted bench comparisons. Error = model / measurement -1, in %.

Shaft	Stiffness	Bench	FEM model	Calculator
P1, regular braid	*EI* (N mm^2)	201.6	+2.6	−3.1
P2, half-load braid	*EI* (N mm^2)	193.4	+9.1	+4.0
P3, diamond braid	*EI* (N mm^2)	200.5	+0.9	−6.9
P1, regular braid	*GJ* (N mm^2)	1418	+7.5	+9.1
P2, half-load braid	*GJ* (N mm^2)	1365	−11.1	+2.2
P3, diamond braid	*GJ* (N mm^2)	868	−20.7	−17.7
T1, coil under braid	*EA* (N)	6740	+8.8	+7.2
Median error (limit 15 %)			**8.8**	**6.9**
Largest error (limit 25 %)			**20.7**	**17.7**

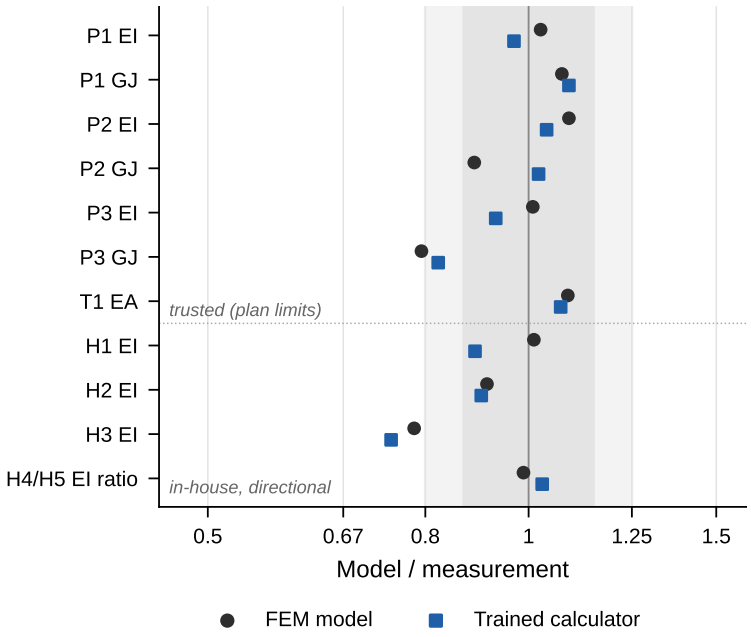

Figure 5. Model / measurement for all bench comparisons. Shaded bands: ±15 % and ±25 %. The plan limits apply to the seven trusted comparisons at the top; the in-house rig shows direction only.

4.5 The calculator reproduces the FEM model

The calculator was scored only on designs that it did not learn from. Four different splits make sure that the score does not depend on near neighbours in the training data (Section 7).

Single braids. On 1,433 single-braid and steerable designs, the median error against the FEM model was 8.7 % in bending, 6.6 % in torsion and 2.1 % in axial stiffness (Fig. 6). The share of designs within 15 % of the FEM model was 66 % in bending, 72 % in torsion and 94 % in axial stiffness (Fig. 7). The score changed by less than one point when whole batches, families or regions of the design space were held out (Table 6).

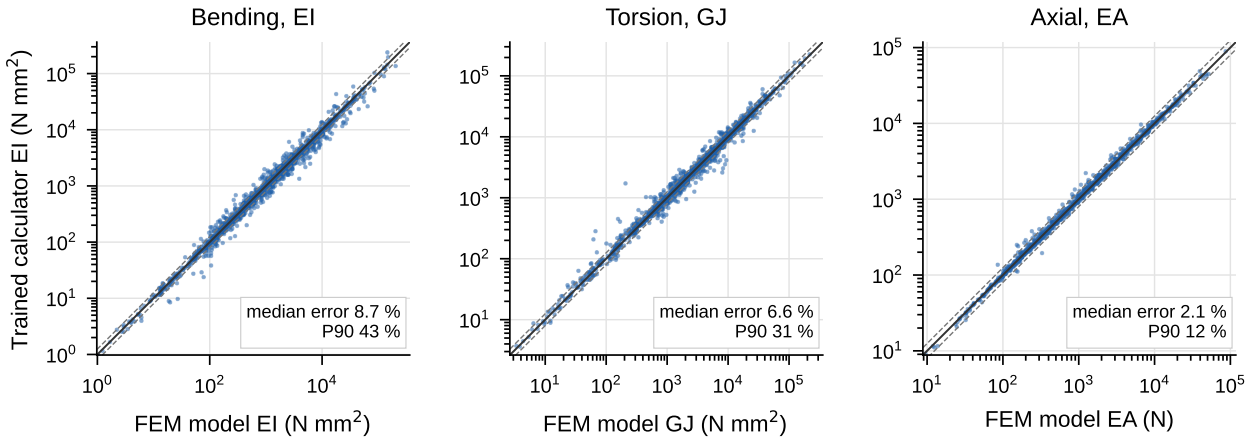

Figure 6. Trained calculator against the FEM model on 1,433 designs. A calculator that did not see the design's batch predicts each design. Solid line: equality; dashed lines: ±25 %.

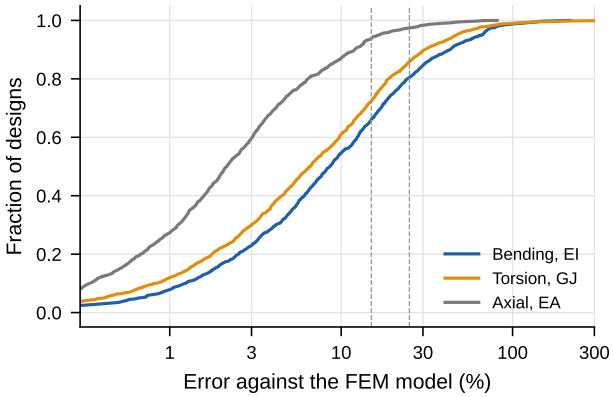

Figure 7. Share of designs below a given error, for the designs of Fig. 6. Dashed lines: 15 % and 25 %.

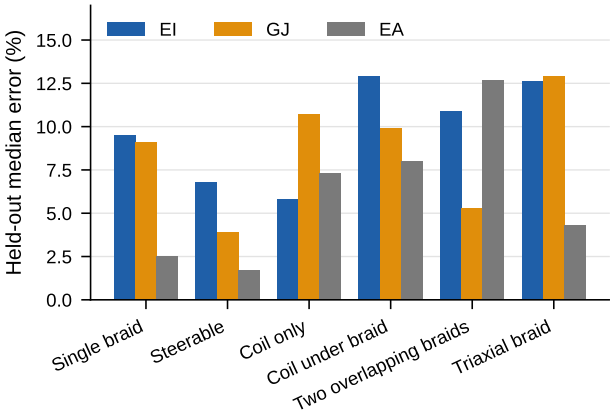

Figure 8. Held-out median error of the calculator against the FEM model, by construction family.

Other families. Coils, coils under braids, two overlapping braids and triaxial braids have median errors of 5.3–12.9 % in bending and torsion and 4.3–12.7 % in axial stiffness (Fig. 8).

The error tail. About one design in ten had an error above 43 % in bending or above 31 % in torsion. The tail is heaviest for less common constructions: braids with three or more filaments per carrier (median 30 % in bending, 24 % in torsion) and micro-catheters below 3 Fr, or 1 mm outer diameter (13 % and 12 %). Mainstream single-filament braids are also about half of the worst tenth.

Table 6. Held-out median error of the calculator against the FEM model (%), with 95 % confidence intervals.

Split	*EI*	*GJ*	*EA*
Single braid (947 designs)			
Random 70/30	8.4 [5.6, 12.3]	7.8 [5.7, 9.9]	2.3 [1.9, 2.8]
Leave one batch out	9.5 [6.2, 13.3]	9.1 [6.8, 11.7]	2.5 [1.7, 3.3]
Leave one family out	9.7 [6.3, 13.6]	9.3 [6.9, 11.7]	2.5 [1.8, 3.4]
Blocked regions	9.2 [6.4, 12.9]	8.5 [7.0, 11.4]	2.5 [1.7, 3.6]
Steerable (486 designs)			
Random 70/30	7.4 [5.2, 11.2]	3.7 [3.2, 5.8]	1.7 [1.2, 3.7]
Leave one batch out	6.8 [5.5, 10.6]	3.9 [2.8, 6.0]	1.7 [1.2, 3.0]
Leave one family out	7.4 [5.2, 10.9]	4.1 [2.9, 8.2]	1.9 [1.5, 3.6]
Blocked regions	7.7 [5.6, 12.9]	4.2 [3.0, 7.1]	2.4 [1.7, 3.8]

Comparing designs. Designers often need to know how two similar designs compare more than they need one absolute value. Twelve base designs covering all six construction families were each varied one variable at a time: braid density, wire width, jacket stiffness (five PEBAX grades from 25D to 72D), wall thickness and coil pitch, at 70, 85, 115 and 130 % of the base value (221 designs). For each pair of a base design and one of its variants, two errors were compared: the absolute error of the variant against the FEM model, and the error in the predicted ratio between the two designs, that is, how much stiffer or softer the variant is than the base. Similar designs share most of the calculator's error, so that error largely cancels when they are compared.

The difference between two similar designs was predicted much more closely than either design alone (Fig. 9, Table 7). The median error of the ratio between two designs was 2.4 % in bending, 1.6 % in torsion and 0.8 % in axial stiffness, against absolute errors of 9.7, 7.7 and 2.4 %; it was the smaller of the two in 72–79 % of the pairs. The closer the two designs, the more of the error cancels: a change of wall thickness kept 0.4 % of error in bending against 8.5 % absolute, while a change of jacket grade, which can halve or double the stiffness, kept 6.7 %. The calculator also moved the stiffness in the right direction on every bending and axial step and on 95 % of the torsion steps; the exceptions are braid-density steps in torsion on one steerable design and one with two overlapping braids, and the softest jacket grade.

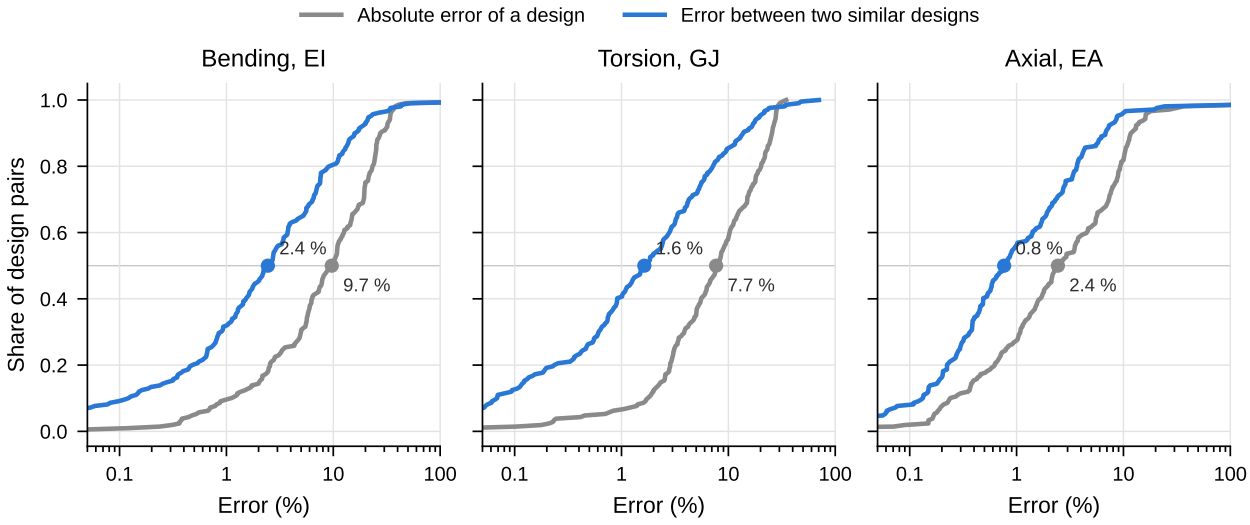

Figure 9. Comparing similar designs. Grey: the absolute error of each design against the FEM model. Blue: the error in the predicted ratio between two similar designs (a base design and a variant with one variable changed). Each curve shows the share of design pairs with an error up to the value on the horizontal axis; dots mark the medians.

Table 7. Comparing similar designs, by variable: median absolute error of a design and median error of the ratio between two similar designs (%).

	Bending, EI		Torsion, GJ		Axial, EA	
Variable changed	Absolute	Between designs	Absolute	Between designs	Absolute	Between designs
Braid density	8.7	3.4	6.2	1.5	1.5	0.8
Wire width	7.9	3.3	7.9	2.4	2.2	0.8
Jacket stiffness	9.6	6.7	9.1	5.9	2.3	1.8
Wall thickness	8.5	0.4	7.4	0.5	3.0	0.2
Coil pitch	12.0	2.3	11.8	0.7	7.8	2.9
All	9.7	2.4	7.7	1.6	2.4	0.8

In short. The calculator gives FEM-quality answers in milliseconds for most designs. When two similar designs are compared, most of its error cancels: the predicted difference between them is typically within 1–3 %. Its error is larger for less common constructions, such as multi-filament braids and micro-catheters below 3 Fr (1 mm outer diameter).

4.6 The frozen calculator on a sealed test

The sealed test is the final, blind check of the calculator against the FEM model. The sealed designs were kept apart from all training and model selection. The calculator was frozen before any sealed result was read, and the sealed set was then scored once, with metrics fixed in advance. The calculator in this release is the frozen one. The sealed set has 400 designs covering all six construction families: 170 single braids, 87 steerable shafts, 59 coils, 64 coils under braids, 10 shafts with two overlapping braids and 10 triaxial-braid shafts. Of the 400, 397 were usable; the FEM solution of the other 3 (all triaxial braids) did not converge.

Result. The median error of the calculator against the FEM model was 7.9 % in bending, 8.3 % in torsion and 2.6 % in axial stiffness (Table 8, Fig. 10). Within 15 % of the FEM model were 74 %, 73 % and 91 % of the designs. Within each family, the calculator ranked the designs in the same order as the FEM model with Kendall $\tau = 0.88$–1.00 in bending and 0.90–0.96 in torsion.

These values agree with the held-out estimates of Section 4.5. For single braids and steerable designs, every sealed median is inside the 95 % interval of the blocked-region estimate (Table 6) or below it. Coils (5.1, 11.1 and 7.0 %) match their development estimates (5.8, 10.7 and 7.3 %), and shafts with two overlapping braids do better than theirs. The seven usable triaxial-braid designs are too few for a firm estimate; their torsion median is 16.6 %. For coil-under-braid designs, torsion (9.5 %) and axial stiffness (6.9 %) agree with their development estimates (9.9 % and 8.0 %), but bending is worse: 18.4 % against 12.9 %. Bending of coil-under-braid shafts is the weakest point of the calculator; treat those values with more caution.

In short. On designs that nobody looked at before the calculator was frozen, the median error was 7.9 % in bending, 8.3 % in torsion and 2.6 % in axial stiffness. The sealed test confirms the development estimates on new designs. The one exception is the bending stiffness of coil-under-braid shafts, where the median error is 18.4 %.

Table 8. Sealed test: error of the frozen calculator against the FEM model (%). The test was scored once, after the freeze. Medians with 95 % confidence intervals.

Measure	*EI*	*GJ*	*EA*
All usable designs (397 of 400)			
Median error	7.9 [6.6, 8.9]	8.3 [6.5, 9.4]	2.6 [2.1, 3.0]
P90	28	37	13
Within 15 % (share of designs)	74	73	91
Within 25 % (share of designs)	87	83	96
Bias (median signed error)	-3.8	+1.1	-0.6
Median error by family			
Steerable (87)	5.2	4.4	1.2
Single braid (170)	9.8	9.7	2.1
Coil under braid (64)	18.4	9.5	6.9
Coil only (59)	5.1	11.1	7.0
Two overlapping braids (10)	7.0	4.1	4.6
Triaxial braid (7)	3.2	16.6	0.7

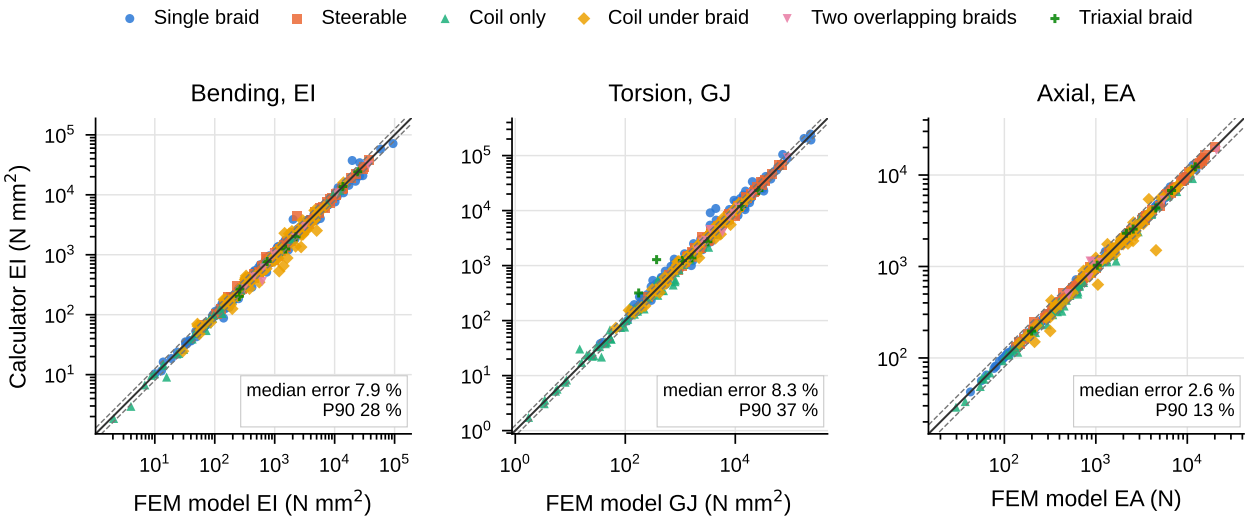

Figure 10. Frozen calculator against the FEM model on the sealed test. Solid line: equality; dashed lines: ±25 %.

4.7 What the error means for a design decision

The section stiffness of a four-way steerable catheter was scaled, and the steering simulation was run (Fig. 11). A 10 % change in *EI* changed the predicted tip angle by about 6 % and the pull-wire force by 4–5 %. A 20 % change in *GJ* had no effect on planar steering. Torsion matters for torque transmission, which this test does not include.

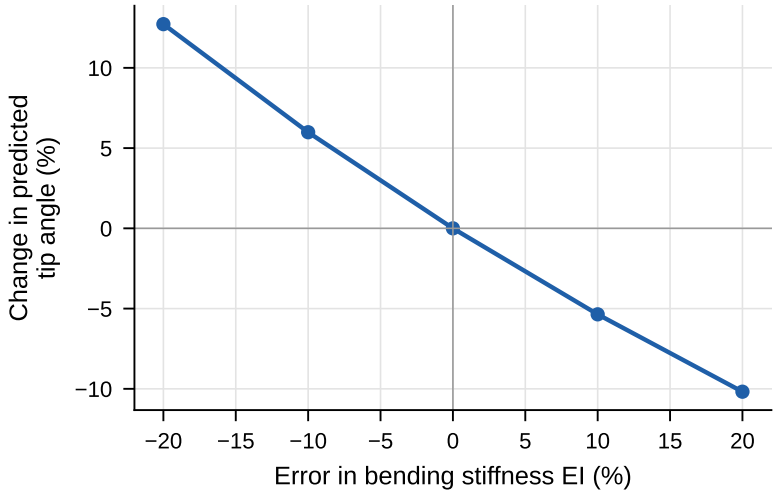

Figure 11. How an error in bending stiffness carries into a steering prediction. Each section's *EI* of a four-way steerable catheter was scaled by the factor on the horizontal axis; the curve is the resulting change in the predicted tip angle.

5 What this means for design work

Accuracy without tuning. Both models meet the plan limits on the trusted bench data, and no constant in either model was fitted to a bench. A tuned model can match a few shafts without capturing the mechanics of other designs. These models earn their agreement from the mechanics.

Generality that can be measured. Because the calculator never sees the bench, its accuracy can be measured on designs it did not learn from. Its score changes little when whole batches, families or regions of the design space are held out. Its transfer to real shafts rests on the evidence for the FEM model: verification, benchmarks and bench agreement without tuning.

How to use the results.

- Use the calculator to screen variants and to compare designs. The difference between two similar designs is predicted more closely than either absolute value.
- Expect a typical error of about 7.9 % in bending, 8.3 % in torsion and 2.6 % in axial stiffness on a new design.
- Take more care with less common constructions: the bending of a coil under a braid, braids with three or more filaments per carrier, and micro-catheters below 3 Fr (1 mm outer diameter).
- Build and test the selected design. The calculator supports design decisions; it does not replace verification tests.

6 Limits and next steps

Limits. The limits are stated openly:

- **No blind bench validation yet.** The trusted bench shafts helped to choose between modelling options during development, so the bench comparison is development evidence. The sealed test is blind, but it checks the calculator against the FEM model, not against the bench.
- **Sealed-test scope.** The 400 sealed designs cover all six construction families, but less common constructions have few designs: the triaxial-braid result rests on seven.
- **Torsion and braid density.** On some steerable designs and designs with two overlapping braids, the calculator does not follow the FEM model's change in torsional stiffness with braid density; compare torsion across braid densities with care there.
- **The calculator inherits the FEM model's error.** It cannot be more accurate than the model it learns from.
- **Ideal construction.** The models do not include voids, unwetted wires or process variation.
- **Datasheet material values.** The models use datasheet moduli, measured on standard test specimens, dry, at room temperature. The polymer in a real shaft can differ: extrusion, orientation and reflow change its properties, and so do the test conditions, for example body temperature (37 °C) against room temperature, or a hydrated against a dry shaft.
- **Bench uncertainty** is not yet carried into the comparison, and a fully resolved 3-D solid reference model is not yet done.
- **Twist level.** The FEM torsion values are at a small twist rate. At large twist, the FEM model predicts lower torsional stiffness; the twist level of the published bench test is not known.

Next steps.

1. **Blind bench campaign (future work).** A campaign is designed but not yet run: shafts will be built and measured without access to the predictions, and the comparison will be published when it is complete.

7 Technical notes

Error metric. For the calculator against the FEM model, the error is $|\ln(\text{calculator}/\text{FEM})|$, given as a percentage. For the bench, the error is model / measurement -1.

Splits. Random splits overstate how well a model generalises [14]. Four were used: random 70/30; leave one batch out (each batch is one sampling campaign); leave one family out; and spatially blocked folds, with the blocks fixed in the declared design domain.

Confidence intervals. Confidence intervals are 95 % BCa intervals [15] from a bootstrap over batches. Doubling the training data changed the errors by less than 0.5 point.

Sealed-test metrics. The metrics were fixed before the sealed set was drawn: the median and P90 of the error, the share of designs within 15 % and 25 %, the bias (median signed error), 95 % BCa confidence intervals from a bootstrap over designs, and the within-family rank agreement (Kendall τ; 1 means the same order as the FEM model).

Pre-registration. The analysis plan, the design domain, the metrics and the criteria were recorded before the results of each phase existed. Reporting follows the separation of development and test data in TRIPOD [16].

Data. The FEM results, the scripts that produced every number and figure, and the analysis plan are held by VirtuCath LLC and are available to reviewers on request.

Document history. Version 1.0, October 2026: first release.

Glossary. **CLT**, classical laminate theory. **FEM**, finite-element method. **GCI**, grid convergence index, an estimate of numerical error. **P90**, the value that 90 % of designs stay below. **PPI**, picks per inch, the braid density. **Verification**: is the model solved correctly? **Validation**: is it the right model for the use?

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